DM 8

A+ Parfait !

trouver tous les polynômes de $\mathbb{R}[X]$ qui vérifient $P \circ P = P^2$

le $\boxed{\text{polynôme nul}}$ est une solution évidente. Supposons $\deg P = n \in \mathbb{N}$

$\deg(P \circ P) = \deg P \times \deg P = (\deg P)^2$

$\deg (P)^2 = 2 \deg P$

$P \circ P = P^2 \Rightarrow (\deg P)^2 = 2 \deg P$

On pose $\deg P = n$

$n^2 = 2n \Leftrightarrow n^2 - 2n = 0 \Leftrightarrow n(n-2) = 0$ soit $n = 0$ ou $n - 2 = 0 \Leftrightarrow n = 2$

Donc $\deg P = 0$ ou $\deg P = 2$

Si $\deg P = 0$, $P = d$ $(\forall d \in \mathbb{R}^*)$ (car $\deg 0 \neq 0$ mais $\deg 0 = -\infty$)

$P \circ P = d$

$P^2 = d^2$

On résoud $d = d^2 \Leftrightarrow d^2 - d = 0 \Leftrightarrow d(d-1) = 0$ $\overbrace{\text{soit } d = 0}^{\text{impossible car } d \in \mathbb{R}^*}$ ou $\boxed{d = 1}$

Si $\deg P = 2$, $P = ax^2 + bx + c$

$P \circ P = a(ax^2+bx+c)^2 + b(ax^2+bx+c) + c$

$= a(ax^2+bx+c)(ax^2+bx+c) + b(ax^2+bx+c) + c$

$= a(a^2x^4 + abx^3 + acx^2 + abx^3 + b^2x^2 + bcx + acx^2 + bcx + c^2) + b(ax^2+bx+c) + c$

$= a^3x^4 + a^2bx^3 + a^2cx^2 + a^2bx^3 + ab^2x^2 + abcx + a^2cx^2 + abcx + ac^2 + abx^2 + b^2x + bc + c$

$= (a^3)x^4 + x^3(2a^2b) + x^2(a^2c + ab^2 + a^2c + ab) + x(abc + abc + b^2) + ac^2 + bc + c$

$= a^3x^4 + 2a^2b\,x^3 + x^2(2a^2c + ab^2 + ab) + x(2abc + b^2) + ac^2 + bc + c$

$P^2 = (ax^2 + bx + c)^2$

$= (ax^2 + bx + c)(ax^2 + bx + c)$

$= a^2x^4 + abx^3 + acx^2 + abx^3 + b^2x^2 + bcx + acx^2 + bcx + c^2$

$= a^2x^4 + 2abx^3 + x^2(2ac + b^2) + x(2bc) + c^2$

$P \circ P = P^2 \Leftrightarrow \begin{cases} a^3 = a^2 \\ 2a^2b = 2ab \\ 2a^2c + ab^2 + ab = 2ac + b^2 \\ 2abc + b^2 = 2bc \\ ac^2 + bc + c = c^2 \end{cases}$ $\begin{cases} a^2(a-1) = 0 \\ 2b = 2b \\ 2c + b^2 + b = 2c + b^2 \\ 0 = 0 \\ c^2 + c = c^2 \end{cases}$ $\begin{cases} \overbrace{a = 0}^{\text{impossible } \deg P = 2} \text{ ou } a = 1 \\ b = 0 \\ c = 0 \end{cases}$

$\boxed{P = X^2}$

3 sol : $P = 0$, $P = 1$, $P = X^2$